THE MARCH MEETING OF THE CHICAGO SECTION.

THE eleventh regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, March 29, 1902, at the University of Chicago. The following fifteen members were present:

Professor Oskar Bolza, Professor D. F. Campbell, Professor L. E. Dickson, Professor Thomas F. Holgate, Dr. H. G. Keppel, Professor Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor S. W. Shattuck, Professor H. E Slaught, Professor E. J. Townsend, Professor L. G. Weld, Professor H. S. White, Professor J. W. A. Young.

A morning and an afternoon session were held, Professor Townsend performing the duties of chairman at the first and Professor Moore, President of the Society, at the second.

At the last Christmas meeting of the Section a committee, composed of Professors Waldo, Bolza and Townsend, was appointed to consider and report a scheme of uniform requirements for the master's degree for candidates who make mathematics their major subject. This committee presented a preliminary report, which was discussed and ordered to be manifolded for the use of members. It is expected that the report will come up for further discussion at the next meeting of the Section.

The following papers were presented at this meeting:

(1) Professor O. STOLZ: "Nachtrag zum Artikel: 'Zur Erklärung der Bogenlänge und des Inhaltes einer krummen Fläche.'"

(2) Mr. A. T. BELL: "The mutual independence of Hilbert's axioms within the various groups."

(3) Professor H. MASCHKE: "On the superosculation of surfaces."

(4) Professor L. G. WELD: "A certain conic connected with the isotomic relation."

(5) Professor OSKAR BOLZA: "Proof of the sufficiency of Jacobi's condition for a permanent sign of the second variation in the so-called isoperimetric problems."

(6) Professor OSKAR BOLZA: "Concerning the isoperimetric problem on a given surface."

(7) Professor L. E. DICKSON: "On the group defined for any given field by the multiplication table of any given finite group."

(8) Professor KURT LAVES: "Some remarkable cases of libration among the small planets of the Hilda type."

(9) Professor E. J. TOWNSEND: "On the interchange of the order of differentiation."

(10) Professor L. E. DICKSON: "Theorems on the residues of multinomial coefficients with respect to a prime modulus"

Mr. Bell was introduced to the Society by Professor Townsend. The paper by Professor Stolz was presented and read by Professor Moore. The following abstracts will serve to characterize the papers:

Professor Stolz gives a comparison of his exposition of the theory of rectification of curves (*Transactions*, volume 3, pages 23–37) with that of C. Jordan (Cours d'Analyse, 2d edition, volume 1, nos. 105–111).

Hilbert in his "Grundlagen der Geometrie" considers the mutual independence of the five groups into which he divides his system of axioms. He also calls attention to the fact that the axioms within the various groups are mutually independent, but merely refers the reader to his lectures on Euclidean geometry (1898–99) where he shows the independence of *particular* axioms from others of the same group. It is the purpose of Mr. Bell's paper to complete this work by constructing the necessary geometry in each case.

Professor Maschke deduces the conditions which a point P of a surface S has to satisfy in order that at P a contact of the third order between S and a general surface of the second order be possible. These conditions, which for every given case reduce to two, are in general given by the vanishing of a differential covariant of the third order. It follows by a simple proof that the developable surfaces and the surfaces of the second order are the only ones for every point of which a surface of second order exists having a contact of the third order at this point. The paper will be published in the *Transactions*.

If L, M, N are the midpoints of the sides a, b, c of a triangle ABC, and if we lay off on a and b, respectively, $LL_1 = \sigma$ and $MM_1 = r\sigma$; also $LL' = -\sigma$ and $MM' = -r\sigma$, then P', determined by AL', BM' is the *isotomic conjugate* of P_1, determined by AL_1, BM_1. Assuming $r = \text{const.}$, Professor Weld shows that the locus of P_1, or of P', is a conic

φ, through A, B, the centroid G of the triangle, and the ex-centroid G_c. This conic reduces to the "chord of contact" form when $r = b/a$; to the C-medial and the side c if $r = -b/a$; to the A-medial and the exterior B-medial, *i. e.*, the line through B parallel to b, if $r = \pm\infty$; to the B-medial and the exterior A-medial if $r = 0$. Chords through isotomic conjugates are concurrent in a point R, lying upon the exterior C-medial; the correspondence between r and R being one to one. The conic φ has the same relation to the triangle ABR as to the triangle ABC when r is any one of the roots of the cubic

$$r^3 - 5\frac{b}{a}r^2 - 5\frac{b^2}{a^2}r + \frac{b^3}{a^3} = 0;$$

i. e., for the three values of r thus determined, ABR and ABC may be appropriately called isotomic reciprocal triangles with respect to φ. One of these values is $-b/a$, for which R coincides with C. The two conjugate values, $(3 \pm 2\sqrt{2})\, b/a$, determine R_a''' and R_b''' upon the exterior C-medial, which points may be called isotomic foci of the given triangle. There are two other pairs of foci, viz., R_b', R_c' upon the exterior A-medial, and R_c'', R_a'' upon the exterior B-medial. These six foci are con-conic.

The proof announced in Professor Bolza's title is based upon an extension of the lemma on linear differential expressions by which Jacobi proves the analogous theorem in the unconditioned problem. It will be published in the *Transactions*.

Professor Bolza's second paper gives a simple proof of the well known theorem that the extremals for the isoperimetric problem on a given surface are curves of constant geodesic curvature. This paper will be published in the *Mathematische Annalen*.

In his first paper Professor Dickson develops for an arbitrary field a theory which generalizes the results of Burnside announced in a paper "On the continuous group defined by any given group of finite order" (*Proceedings of the London Mathematical Society*, volume 29, 1898). His proofs by means of the Lie theory are replaced by rational operations valid for any field F. In particular, by taking for F the various Galois fields, we obtain a doubly infinite system of finite groups corresponding to each given finite

group. The results find application in the problem to represent a given finite group as a linear group in a given field upon the smallest number of variables.

Laplace has shown that libration in longitude will occur under the following condition: If θ be an angle of the character of the longitude in the orbit, it is defined by the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{m'}{2} h^2 \sin\theta, \tag{1}$$

where h^2 is a function of certain elements of the perturbed and perturbing planets, m' the mass of the perturbing planet. Calling c the constant of integration, we obtain from (1)

$$h\sqrt{m'}dt = \frac{d\theta}{\sqrt{\cos\theta + c}}. \tag{2}$$

The condition for libration is then $c^2 < 1$; θ will then oscillate between two finite values. Determining the value of c from the osculating elements of a planet, it can be shown, that this condition may be written as follows:

$$[in_1 - (i+1)n'] < h \cdot \cos\frac{\theta_1}{2}\sqrt{2m'},$$

for all planets which are of such a type that their mean daily motion is to that of Jupiter as $i+1 : i$, where i is an integer. In the foregoing formula n_1 represents an osculating value of the mean daily motion of the perturbed planet, n' the mean daily motion of Jupiter; θ_1 is a special value of θ. Tisserand, in searching for an explanation of his "lacunes" in the ring of planetoids seized upon the possibility of Jupiter's effecting librations among the planetoids at the places where these "lacunes" take place. He was not able to discover librational effects in these cases. It is a curious result that appears from Professor Laves's paper that libration takes place at those very places $n = 450''$ and $n = 400''$ where no "lacunes" occur.

Professor Townsend discussed the conditions under which the relation

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

is valid. It is usual to show that this holds at any point in

whose neighborhood

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \left(\text{or } \frac{\partial^2 f}{\partial y \partial x} \right)$$

exist and are continuous in the two variables x, y together. In this paper it is shown that a narrower condition is sufficient, namely when

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ and } \frac{\partial^2 f}{\partial x \partial y} \left(\text{or } \frac{\partial^2 f}{\partial y \partial x} \right)$$

exist and are continuous in x alone and in y alone. When these conditions exist, the points are everywhere dense upon any line $x =$ constant, or $y =$ constant, in which $\frac{\partial^2 f}{\partial x \partial y}$ is a continuous function of both variables together. It follows that at these points $\frac{\partial^2 f}{\partial y \partial x}$ also exists and equals $\frac{\partial^2 f}{\partial x \partial y}$. A necessary and sufficient condition is then developed for the interchange of the order of limits in general. By the application of this theorem, the existence and continuity of

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \left(\text{or } \frac{\partial^2 f}{\partial y \partial x} \right)$$

in each variable separately is shown to be a sufficient condition for the interchange of the order of differentiation.

Professor Dickson shows in his second paper that the residue, modulo p, of the sum of certain general sets of multinomial coefficients is either zero or a single reduced multinomial coefficient. This result is a generalization of a theorem on binomial coefficients due to Glaisher (*Quarterly Journal of Mathematics*, volume 30 (1899), pages 361–383). The proof depends on two theorems announced by the author in the *Annals of Mathematics*, 1st series, volume 11 (1897), pages 75–76.

THOMAS F. HOLGATE,
Secretary of the Section.

CONCERNING THE ANGLES AND THE ANGULAR DETERMINATION OF PLANES IN 4-SPACE.

BY DR. C. J. KEYSER.

(Read before the American Mathematical Society, April 26, 1902.)

I. *Introductory: Angles of Two Planes.*

IT has been established by Jordan * that, if two linear point spaces S_α and S_α', each of $n-\alpha$ dimensions, in a linear point space of n dimensions, have one and but one point in common, S_α and S_α' form α angles. In the special case where $n=4$ and $\alpha=2$, S and S' denote two planes of 4-space, and, as two planes in 4-space have one and in general but one common point, it follows that two planes of 4-space form two angles. That such is the case can be readily seen independently of the general theorem mentioned, in either of the following reciprocal ways: Two planes π and π' may be conceived as two flat pencils of lines having the common point P of the planes as common vertex. Of all the angles formed by lines of the one pencil with lines of the other there is a least angle θ, and this is not zero since by hypothesis the pencils have no line in common. Denote by π'' the plane determined by the lines l and l' where the latter belong respectively to π and π' and form the angle θ. As the planes π' and π'' have the line l' in common, they form an ordinary angle α. In order to bring π and π' into coincidence, it is sufficient to turn l' in π'' through the angle θ and then, as l and l' now coincide and π and π' have a common line, to rotate π or π' through the ordinary angle ω formed by π with π' in its new position. Reciprocally, π and π' may be regarded as pencils of lineoids † (ordinary 3-spaces). Either plane will accordingly be determined by any pair of its generating lineoids. *i.e.*, the lineoids enveloping or containing it. The planes π and π' have no common lineoid, for otherwise they would have, contrary to hypothesis, a common line. Accordingly there is a minimal angle φ, not zero, in the assemblage of angles formed by lineoids of the one plane with those of the other. Denote by π'' the plane determined by L and L', where L and L' are lineoids of π and π' respectively and

* Jordan: "Essai sur la géométrie à n dimensions," *Bull. de la Soc. Math. de France*, vol. 3, p. 129.

† Cf. Cole: "On rotations in space of four dimensions," *Amer. Jour. of Math.*, vol. 12.

form the angle φ. To bring π and π' into coincidence it is sufficient to rotate L' about π'' through the angle φ, which will cause π and π' to lie in a same lineoid L and so to intersect in a line, and then to turn π' or π about this line through the angle γ formed by π with π' in its new position.

Hoppe* has shown that $\theta = \varphi$ and $\omega = \gamma$. The angles between two planes of 4-space may, therefore, receive the following equivalent reciprocal definitions: (1) The minimal angle θ between a lineoid L of the one plane and a lineoid L' of the other, and the angle ω formed by the planes when they have been made collineoidal by the rotation of L or L' about their common plane through the angle θ; (2) the minimal angle θ between a line l of the one pencil (plane) and a line l' of the other, and the angle ω formed by the planes when they have been made collinear by rotating l or l' in their common plane through the angle θ.

A point (of 4-space) regarded as a plenum of planes is 4-dimensional. The plane geometry of this assemblage is of considerable interest both on its own account and on account of its analytical similarity to several other geometric theories, such as the Plücker line geometry, the sphere geometry of ordinary space, and particularly the geometry of the sphere surface which arises on employing as element the pencil of circles, *i. e.*, an assemblage of circles of the sphere which have a pair of points in common. It is the object of this note to construct by means of the angles θ and ω a system of homogeneous coördinates, analogous to the Plücker coördinates, for the plane regarded as element of a 4-space point.

II. Homogeneous Coördinates of the Plane.

Without loss of generality we may choose the point in question for origin of rectangular point coördinates x_i $(i = 1, 2, 3, 4)$. Assuming four mutually perpendicular lineoids, $x_i = 0$, for coördinate lineoids, these will determine six planes and four lines.

Any plane π of the assemblage of generating planes of the origin may be determined by a pair of lineoids

$$(1) \qquad \begin{cases} \Sigma a_i x_i = 0, \\ \Sigma b_i x_i = 0, \end{cases} \qquad (i = 1, 2, 2, 4).$$

* Hoppe: "Ueber die Stellung der Ebene in der Vierdimensionengeometrie," *Archiv d. Math. u. Physik.*, vol. 68.

The plane is equally determined by any two lineoids of the pencil

$$\sum(\lambda a_i + \mu b_i)x_i = 0 \tag{2}$$

of generating lineoids of π, and, in particular, by any pair of the special lineoids corresponding to the parametric values

$$\mu : \lambda = -a_i : b_i \qquad (i = 1, 2, 3, 4).$$

The equations of these lineoids are

$$\left\{\begin{aligned} 0.x_1 + p_{12}x_2 + p_{13}x_3 + p_{14}x_4 &= 0, \\ -p_{12}x_1 + 0.x_2 + p_{23}x_3 + p_{24}x_4 &= 0, \\ -p_{13}x_1 - p_{23}x_2 + 0.x_3 + p_{34}x_4 &= 0, \\ -p_{14}x_1 - p_{24}x_2 - p_{34}x_3 + 0.x_4 &= 0, \end{aligned}\right. \tag{3}$$

where $p_{ik} \equiv a_i b_k - b_i a_k$.

These lineoids are geometrically distinguished among the generating lineoids of π by the fact that they each contain a coördinate line. The quantities p entering any pair of the equations (3) furnish four ratios, which are sufficient to determine the plane. The entire six p's furnish five ratios, which are, however, equivalent to four, by virtue of the identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \tag{4}$$

The functions p satisfy, moreover, the relations

$$p_{ii} = 0, \quad p_{ik} = -p_{ki}. \tag{5}$$

The ratios are independent of the choice of defining lineoids, for if π be supposed given by the lineoids, $\lambda_1 a_i + \mu_1 b_i$ and $\lambda_2 a_i + \mu_2 b_i$, the corresponding quantities p_{ik}' will be

$$p_{ik}' = (\lambda_1\mu_2 - \lambda_2\mu_1)p_{ik}.$$

It can be readily shown that, conversely, any six functions p_{ik} that satisfy the relations (4) and (5) will, if employed as in equations (3), determine four lineoids, enveloping, all of them, a same plane of the assemblage in question.

Accordingly, we may retain for coördinates of the plane, regarded as envelope of lineoids, six variables p_{ik} which are subject to the relations (4) and (5). It will be sufficient to mention in passing that there is a reciprocal system of coör-

dinates for the plane conceived as a pencil of lines. These coördinates will be proportional to the six coefficients q_{ik} of the four equations representing the four lines which the plane has in common with the coördinate lineoids. These two coördinate systems correspond to and are formally identical with the Strahlencoordinaten and Axencoordinaten of Plücker.

III. *Trigonometric Interpretation of the Coördinates p_{ik}'.*

Suppose two planes π and π' to be given by the two pairs of lineoids

$$(6) \qquad \begin{cases} \Sigma\xi_i x_i = 0, & \Sigma\xi_i' x_i = 0, \\ \Sigma\eta_i x_i = 0, & \Sigma\eta' x_i = 0, \end{cases} \qquad (i = 1, 2, 3, 4).$$

If we assume, which may be done without loss of generality, that the equations (6) are in the normal form and also that the lineoids ξ_i and η_i as also ξ_i' and η_i' are perpendicular, then

$$(7) \qquad \Sigma\xi_i^2 = \Sigma\eta_i^2 = \Sigma\xi_i'^2 = \Sigma\eta_i'^2 = 1,$$

$$(8) \qquad \Sigma\xi_i\eta_i = \Sigma\xi_i'\eta_i' = 0.$$

In the paper already cited Hoppe shows that $\cos\theta$ and $\cos\omega$, where θ and ω are the angles of π and π', are connected by the relation

$$(9) \quad \cos\theta\cos\omega = \Sigma\xi_i\xi_i'\Sigma\eta_i\eta_i' - \Sigma\xi_i\eta_i'\Sigma\eta_i\xi_i' \quad (i = 1, 2, 3, 4).$$

We have $\theta = \omega = 90°$, and then π and π' are said to be *absolutely* perpendicular, when and only when

$$(10) \qquad \Sigma\xi_i\xi_i' = \Sigma\eta_i\eta_i' = \Sigma\xi_i\eta_i' = \Sigma\eta_i\xi_i' = 0,$$

i. e., when and only when every generating lineoid of the one plane is perpendicular to every generating lineoid of the other. It follows that the planes of each of the pairs (π_{12}, π_{34}), (π_{13}, π_{24}), (π_{14}, π_{23}) of coördinate planes, where $\pi_{\alpha\beta}$ stands for the plane determined by the coördinate lineoids $x_\alpha = 0$, $x_\beta = 0$, are absolutely perpendicular. If, now, we denote by $\theta_{\alpha\beta}$ and $\omega_{\alpha\beta}$ the angles of the plane $\pi_{\alpha\beta}$ with the plane π

$$(11) \qquad \begin{cases} \Sigma a_i x_i = 0, \\ \Sigma b_i x_i = 0, \end{cases}$$

where

$$\Sigma a_i b_i = 0, \quad \Sigma a_i^2 = \Sigma b_i^2 = 1,$$

we have the relations

$$(12)\qquad \begin{cases} \cos\theta_{12}\cos\omega_{12} = p_{12}, & \cos\theta_{34}\cos\omega_{34} = p_{34}, \\ \cos\theta_{13}\cos\omega_{13} = p_{13}, & \cos\theta_{24}\cos\omega_{24} = p_{24}, \\ \cos\theta_{14}\cos\omega_{14} = p_{14}, & \cos\theta_{23}\cos\omega_{23} = p_{23}, \end{cases}$$

where $p_{\alpha\beta} = a_\alpha b_\beta - b_\alpha a_\beta$, and the relation

$$(13)\qquad \begin{aligned} &\cos\theta_{12}\cos\omega_{12}\cos\theta_{34}\cos\omega_{34} - \cos\theta_{13}\cos\omega_{13}\cos\theta_{24}\cos\omega_{24} \\ &\quad + \cos\theta_{14}\cos\omega_{14}\cos\theta_{23}\cos\omega_{23} \equiv 0. \end{aligned}$$

It remains to determine the changes which the expressions for the cosine products undergo in case the assumption that the defining lineoids (6) are perpendicular is abandoned. In this case we may suppose π defined by the pair of perpendicular lineoids,

$$(14)\qquad \begin{cases} \Sigma(\xi_i + \eta_i)x_i = 0, \\ \Sigma(\xi_i - \eta_i)x_i = 0, \end{cases}$$

and π' by the pair,

$$(15)\qquad \begin{cases} \Sigma(\xi_i' + \eta_i')x_i = 0, \\ \Sigma(\xi_i' - \eta_i')x_i = 0. \end{cases}$$

If now these equations be written in the normal form, the expression for $\cos\theta\cos\omega$ becomes after reduction

$$(16)\qquad \frac{\Sigma\xi_i\xi_i'\Sigma\eta_i\eta_i' - \Sigma\xi_i\eta_i'\Sigma\eta_i\xi_i'}{\sqrt{\{1 - (\Sigma\xi_i\eta_i)^2\}\{1 - (\Sigma\xi_i'\eta_i')^2\}}}.$$

Denoting the angles between the lineoids ξ_i and η_i, ξ_i' and η_i', $\cdots$, by the symbols (ξ, η), (ξ', η'), $\cdots$, we may write

$$(17)\quad \cos\theta\cos\omega = \frac{\cos(\xi, \xi')\cos(\eta, \eta') - \cos(\xi, \eta')\cos(\eta, \xi')}{\sin(\xi, \eta)\sin(\xi', \eta')}.$$

If the lineoids a_i and b_i of equations (11) are supposed to be non-perpendicular, equations (12) assume the form

$$(18)\qquad \cos\theta_{\alpha\beta}\cos\omega_{\alpha\beta} = p_{\alpha\beta} : \sin(a, b).$$

Inasmuch, however, as only the ratios of the p's are essential, $\sin(a, b)$ may be disregarded or absorbed in $p_{\alpha\beta}$.

It appears, therefore, that in studying the geometry of planes through a point of 4-space, it is possible, instead

of referring the plane to lines or lineoids as coördinate elements, to determine the plane by reference to planes, as follows: *if from the plane assemblage in question six planes* $\pi_1, \cdots, \pi_6$ *be chosen such that they furnish three pairs* $\pi_1\pi_6$, $\pi_2\pi_5$, $\pi_3\pi_4$ *of absolutely perpendicular planes, then we may take for homogeneous plane coördinates of a plane* π *the six cosine products* $\cos\theta_\alpha \cos\omega_\alpha$, *where* θ_α *and* ω_α *are the angles of* π *with the coördinate plane* π_α.

The six coördinate planes intersect an arbitrary lineoid of 4-space in the six edges of a tetraedron, while the arbitrary plane π cuts from the lineoid an equally arbitrary line. The angles θ and ω correspond to the distances and angles of the line with the tetraedral lines. The corresponding interpretation of the Plücker coördinates would accordingly be one in terms of these distances and angles.

We will close this note with the necessary and sufficient condition that two planes $(\theta_\alpha, \omega_\alpha)$ and $(\theta_\alpha', \omega_\alpha')$ shall have a common line. It is

$$\begin{aligned}(19)\quad &\cos\theta_1\cos\theta_6'\cos\omega_1\cos\omega_6' + \cos\theta_1'\cos\theta_6\cos\omega_1'\cos\omega_6\\ &- \cos\theta_2\cos\theta_5'\cos\omega_2\cos\omega_5' - \cos\theta_2'\cos\theta_5\cos\omega_2'\cos\omega_5\\ &+ \cos\theta_3\cos\theta_4'\cos\omega_3\cos\omega_4 + \cos\theta_3'\cos\theta_4\cos\omega_3'\cos\omega_4 = 0.\end{aligned}$$

COLUMBIA UNIVERSITY,
April, 1902.

NOTE ON THE SUFFICIENT CONDITIONS FOR AN ANALYTIC FUNCTION.

BY MR. D. R. CURTISS.

(Read before the American Mathematical Society, April 26, 1902.)

SINCE the publication of Goursat's proof * that a function of the complex variable z possessing a derivative at each point of a two-dimensional region T in which it is single-valued must necessarily have a continuous derivative throughout that region, the question has arisen whether the sufficient conditions for an analytic function, stated in terms of the partial derivatives of the real and pure imaginary parts of the function, may not be reduced to simpler terms. These conditions are ordinarily given as follows:

$w = u(x, y) + iv(x, y)$ is an analytic function of the complex variable $z = x + iy$ at each point of a region T of the z-plane if throughout T

* *Trans. Am. Math. Soc.*, Vol. 1 (1900), No. 1, p. 14.

1) u and v are single-valued, continuous functions of the independent real variables x and y;

2) $$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}$$

exist;

3) the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied;

4) these four partial derivatives are continuous functions of the independent variables x and y.

That condition 4) demands more than is necessary to prove the existence of a derivative of w has been noted by Harnack,* and later by Stolz,† both writers showing that in its place may be substituted the requirement that u and v each possess a total differential. So far as I have been able to ascertain, no further substantial simplification of condition 4) has ever been published.

The result which I have obtained is that *conditions* 1), 2), 3), *together with the continuity of* ONE *of the four partial derivatives of* 2), *are sufficient for the existence of a derivative* $\frac{dw}{dz}$ *and hence, by Goursat's theorem, are sufficient conditions that* w *be an analytic function of* z *throughout the region* T.

Let $z_0 = x_0 + iy_0$ be any point in T; and let $w = u(x, y) + iv(x, y)$ be a function whose real and imaginary parts, u and v, satisfy conditions 1), 2), and 3). Further, assume $\frac{\partial u}{\partial x}$ continuous in both x and y at each point of T. Then at the point (x_0, y_0) we have

$$\Delta u = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) + u(x_0, y_0 + \Delta y) - u(x_0, y_0)$$

$$= \frac{\partial}{\partial x} u(x_0 + \theta_1 \Delta x, y_0 + \Delta y)\Delta x + \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}\Delta y$$

$$(0 < \theta_1 < 1).$$

On the hypothesis that $\frac{\partial u}{\partial x}$ is continuous and $\frac{\partial u}{\partial y}$ exists, we have then

* Differential- und Integral-Rechnung, p. 152.

† Differential- und Integral-Rechnung, vol. II, p. 82.

$$\Delta u = \left(\frac{\partial u}{\partial x} + \zeta_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \zeta_2\right)\Delta y\Big|_{x_0, y_0},$$

where ζ_1 and ζ_2 approach the limit zero when Δx and Δy, independently of each other, approach the limit zero.

From equations 3), $\dfrac{\partial v}{\partial y}$ must be continuous, hence we may write

$$\Delta v = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0 + \Delta x, y_0) + v(x_0 + \Delta x, y_0) - v(x_0, y_0),$$

and by reasoning as before,

$$\Delta v = \left(\frac{\partial v}{\partial y} + \zeta_3\right)\Delta y + \left(\frac{\partial v}{\partial x} + \zeta_4\right)\Delta x\Big|_{x_0, y_0},$$

where ζ_3 and ζ_4 approach the limit zero when Δx and Δy, independently of each other, approach the limit zero.

Using these expressions for Δu and Δv, we have the following form for the difference-quotient $\dfrac{\Delta w}{\Delta z}$;

$$\frac{\Delta w}{\Delta z}\Big|_{z_0} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \frac{\left(\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}\right)\Delta x + \left(\dfrac{\partial u}{\partial y} + i\dfrac{\partial v}{\partial y}\right)\Delta y}{\Delta x + i\Delta y} + \eta,$$

where the derivatives are formed for the point (x_0, y_0), and η approaches the limit zero with Δx and Δy.

The first term on the right-hand side can, with the aid of condition (3), be thrown into the form $\dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$. Hence $\dfrac{\Delta w}{\Delta z}$ approaches a limit when Δz approaches zero, and

$$\lim_{\Delta z = 0} \frac{\Delta w}{\Delta z} = \frac{dw}{dz}\Big|_{z_0} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{x_0, y_0}$$

We have thus proved the existence of a derivative, assuming the continuity of only one of the partial derivatives of (2). From the nature of the proof it is seen that this partial derivative might have been any one of the four.

The question still remains unanswered as to whether we may not dispense with *any* assumption concerning the continuity of the partial derivatives of u and v.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
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SCHEFFERS'S THEORY OF SURFACES.

Einführung in die Theorie der Flächen. Von Dr. GEORG SCHEFFERS. Der "Anwendung der Integral- und Differential-Rechnung auf Geometrie" zweiter Band. Leipzig, Veit & Comp., 1902. Pp. x + 518.

THERE will naturally be a considerable divergence of opinion on the part of mathematicians both as to what such a volume as the one before us should contain, and, from a pedagogical standpoint, as to how the subject matter should be presented. It may be stated at once that in the opinion of the present writer the author has accomplished with marked success, on the whole, the difficult task of placing before the beginner in a clear and readable form an outline of the vast theory of surfaces and surface curves.

Among the advantages of the work which recommend it powerfully to the student taking up this branch of mathematics for the first time, is the almost entire absence of unaccustomed symbols and symbolic operators, the free use of which adds so much to the difficulty of reading Bianchi, or Stahl-Kommerell, for example. Bearing in mind that u, v are the parametric coördinates of a point on the surface, that E, F, G and L, M, N are the fundamental magnitudes, that X, Y, Z are the direction cosines of the surface normal, R_1, R_2 the principal radii of curvature, and finally that K, H represent respectively the Gaussian measure of curvature, and the mean curvature, the student will hardly be at loss to read understandingly any chapter in the book—as far at least as the symbolism is concerned.

The author has, very wisely I think, avoided presenting the theory of surfaces as the theory of invariance of two quadratic differential forms. While there are undoubtedly advantages attached to the latter method of presenting the subject, such as the brevity and elegance of most of the formulas developed, yet this method never appeals to the beginner as being a natural one; and the gain in brevity is apt to be more than compensated for by a loss in clearness. Moreover, to obtain a proper insight into the subject, the student should familiarize himself with both methods of treatment, taking the easier and more natural one first.

Another striking advantage for the beginner is that in this volume, as in the preceding one on space curves, every theorem given, pertaining to the subject, is developed in

full from the analytical standpoint, geometrical proofs being appended only in the cases in which they add materially to an understanding of the scope of a theorem, or where they are of especial historical interest. Even when a theorem is given which belongs to an entirely different branch of mathematics, as the one concerning the integration of the so-called unconditionally integrable total differential equations, pages 321–331, the necessary proof is usually given.

Also in this volume, as in the preceding one, the author has taken great care to deduce and formulate all theorems in such manner as to cover the cases in which imaginaries are involved.

If objections are to be made at all to this excellent work, they must, in my opinion, be of a pedagogical nature: and even these are largely matters of individual taste. I will mention, however, a few points in which it seems to me that the author might have added to the usefulness of the book.

Although the volume is already quite bulky, it seems to me unfortunate that it did not open with a separate chapter on the curvature of surfaces from the standpoint of Monge, assuming the surface to be given by an equation of the form $z = f(x, y)$. The author occupies from the beginning the standpoint of Gauss; that is, he assumes the surface to be given by three equations connecting the cartesian coördinates x, y, z with the two parameters u, v. Monge's treatment of the subject is of great historical interest; and the symbols, and many of the formulas occurring, are perfectly familiar to a student who has read only the calculus. It is true that the classic formulas of Monge and Euler are collected in a Table (Tafel XIII.) at the end of the book; but a reader familiar with the treatises of Laurent, Picard, and others, will note with regret the absence from the body of the book of a consistent though brief development of the theory of curvature from Monge's standpoint; and the loss to the beginner is, in my opinion, a serious one.

It also appears regrettable that a chapter was not given on families of surfaces, and their defining partial differential equations. This would seem to be a proper subject to be incorporated in a work on the "Applications of the differential and integral calculus to geometry," and would undoubtedly have added much to the interest of the book. Moreover, the author has shown himself, in several of Lie's works edited by him, to be well fitted to treat this subject in an attractive and instructive manner.

We note with less regret the absence of any chapter on curvilinear coördinates in space.

Another minor objection is the same as that which was made to the volume on space curves: that is, that while many important problems, such as those illustrating the theory of map making, the congruence of surfaces, etc., are worked out more fully than is usual, and with remarkable care, there are no collections of problems for the student at the ends of the chapters—a very desirable feature of most English and of many French works.

I shall now give a general outline of the matter presented; and speak in some detail of a few sections of the work.

As presented by the author, the subject matter is arranged in four divisions: I, The Arc Element of the Surface; II, Curvature; III, The Fundamental Equations of the Theory of Surfaces; IV, Surface Curves.

The student who is familiar with the sections of Volume I on parametric coördinates in the plane, will have no difficulty in reading the opening sections of this volume, in which, besides other introductory matter, the quadratic differential form,

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

is deduced for the square of the arc element, and the geometrical meaning of the fundamental magnitudes of the first order E, F, G is explained. The author proceeds, in §§ 5, 6 to the discussion of the problem of the point representation of one surface upon another, the special case considered being that in which areas are invariant. This is a generalization of the corresponding discussion in § 8, Part I, Volume I; and it is in this connection that several interesting examples of the older methods of map making—those in which areas are invariant—are fully worked out.

The definition and determination of isothermal systems on a surface occupy §§ 7, 8; and here again § 7 is practically the same in form as the corresponding section in Volume I. The author's method for introducing and determining isothermal systems on a surface seems unnecessarily indirect. He first establishes the condition that the parameter curves $u = \text{const.}$, $v = \text{const.}$, shall divide the surface into a network of infinitesimal rhombi, and then that the diagonals of the rhombi shall meet at right angles. These conditions are satisfied, that is the parameter curves form an isothermal system, when the square of the arc element has the form

$$ds^2 = \Omega(u, v)(du^2 + dv^2). \qquad (1)$$

He then proceeds to show that all isothermal systems on a given surface can be found when the differential equation

$ds^2 = 0$ of the minimal curves of the surface can be integrated.

It would be more satisfactory, in my opinion, to present the matter, as for example Bianchi does (Differential Geometry, page 70); that is, show first that the square of the arc element can be thrown into the form (1), when the differential equation $ds^2 = 0$ can be integrated. It then follows at once that $u = \text{const.}$, $v = \text{const.}$ form an isothermal system. Also, if desired, the proof that (1) expresses the necessary as well as the sufficient condition that the parameter curves form an isothermal system can be added in a few lines, as in Darboux, Leçons sur la théorie générale des surfaces, volume I, page 146.

In § 9 we have an excellent exposition of the theory of the conform representation of one surface upon another; and the general theory is illustrated in § 10 by the example of the conform representation of a sphere upon a plane. The important theorem is proved, that the stereographic projections of a given sphere upon a plane are the only conform representations of the sphere upon the plane, by means of which all circles of the sphere appear as circles on the plane. The section closes with a discussion of the theory of Mercator's projection.

The last section of Part I contains a brief outline of the theory of the general point representation of one surface upon another, including Tissot's theorem that in the case of any not conform point representation of one real surface upon another, there is in general one and only one orthogonal system of curves on the one surface which corresponds to an orthogonal system on the other surface. Attention is also drawn to Lie's note (*Mathematische Annalen*, volume 20) in which he shows that this theorem does not always hold for imaginary point representations.

This section is a particularly valuable one, as the problem of the general point representation of one surface upon another is not usually discussed in the text-books.

Part II deals with the theory of curvature of surfaces. The contents of the various sections of this part will be sufficiently indicated by saying that they cover the general topics usually discussed in this connection; namely, conjugate curve systems on a surface; asymptotic curves; lines of curvature; spherical representation of a given surface, after Gauss; minimal surfaces. We note that the author completes in an essential feature the discussion of the curvature of a plane normal section, as it is given in the other current text-books. The curvature, at a point (u, v)

on a given surface, of a plane normal section through (u, v), is represented by

$$\frac{1}{R} = \frac{L + 2Mk + Nk^2}{E + 2Fk + Gk^2}, \tag{2}$$

where k is written for $dv : du$.

The two cases usually considered here are (i) that the right hand member of (2) is a quadratic fractional function of k, the case which occurs at any ordinary point on a real surface; and (ii) that the right hand member of (2) does not actually contain k, that is, that

$$L : M : N = E : F : G, \tag{3}$$

or that the point (u, v) is an umbilic. As is well known, if these conditions are satisfied at every point on a given surface, the surface is either a plane or a sphere.

But there is a third possibility: that is, the numerator and denominator of the right hand member of (2) may have a common factor which is linear in terms of k, in which case (2) can be written

$$\frac{1}{R} = \frac{\alpha k + \beta}{\gamma k + \delta}. \tag{4}$$

This case, which is of interest and importance, seems to have been first discussed by Stäckel, *Leipziger Berichte*, 1896. At a point (u, v) on a given real surface, where (4) is satisfied, there cannot exist principal radii of curvature, since, by (4), $\frac{1}{R}$ has no finite maximum or minimum at that point. An example of a real surface with such a point is the following: We construct in each plane through the z-axis the circle which is tangent to the xy-plane at the origin, and of which the ordinate, $z = R$, of the center, is a linear fractional function of the tangent of the angle which the plane of the circle makes with the zx-plane. The locus of the ∞^1 circles constructed in this manner is a real surface; and the origin is clearly a point of the required nature on that surface.

If the condition (4) is satisfied at every point on a surface, Euler's theory of curvature does not hold for the surface. A surface of this nature can be shown to be imaginary, and to contain one system of minimal straight lines, and one system of minimal curves. The surfaces for which the conditions (3) hold at every point, contain, it will be

remembered, two distinct families of minimal straight lines.

In §13 the author inserts a portion of the theory of the ruled surface, and develops the theory of curvature of the ruled surface as an example of the application of the formulas deduced for the general theory of curvature. It would have been better, in my opinion, to devote a division of the book to a consecutive development of the theory of the straight line system—as is done by Laurent and Jordan, for example—deducing thence in outline both the theory of the line complex, which is so intimately connected with that of the curvature of surfaces, and the theory of the ruled surface. At present the reader has to look through several sections of both Volume I and Volume II to find what is said of the ruled surface, or of the line complex.

In the example, page 183, of the asymptotic curves of a ruled surface, the author shows in the usual manner that the curvilinear asymptotic curves of a ruled surface are defined by a Riccati equation

$$\frac{dv}{du} = A(u) + B(u)v + C(u)v^2;$$

but, contrary to his usual custom, he does not complete the discussion by giving Bonnet's theorem, that the general solution of this equation can be found, when one particular solution is known.

Although the author's discussion of the theory of the lines of curvature of a surface is quite comprehensive, he omits to state that the number of lines of curvature through an umbilic is generally finite, and to show how to obtain the differential equation defining the directions of the lines of curvature through such a point.

The above objections are, however, trivial in comparison with the general excellence of the treatment of the subjects covered by Part II.

Part III, which is considerably more difficult for the beginner than the other divisions of the book, deals primarily with the derivation of the three fundamental partial differential equations of a surface: that is, the three equations connecting the fundamental magnitudes E, F, G, L, M, N with their derivatives. The principal object of this Part is to show that when any six functions E, F, G, L, M, N of u and v satisfy the three fundamental equations, then E, F, G, L, M, N can be considered the six fundamental magnitudes of a certain surface,

and that they do, in fact, define the surface except as regards its position in space. The proof of this great theorem, which is given in several consecutive steps, and which occupies some thirty pages of the book, is, in my opinion, decidedly to be preferred for a beginner to the comparatively compressed proof given by Stahl-Kommerell, or Bianchi. The author recommends, however, that these pages be omitted on a first reading of the volume.

In § 6 of this Part we have the derivation of all differential invariants of a surface under the movements in space

$$x_1 = \alpha_1 x + \alpha_2 y + \alpha_3 z + a,$$
$$y_1 = \beta_1 x + \beta_2 y + \beta_3 z + b,$$
$$z_1 = \gamma_1 x + \gamma_2 y + \gamma_3 z + c.$$

These differential invariants, which are, of course, of prime importance in the theory of congruence of surfaces, are easily seen to be functions of the fundamental magnitudes E, F, G, L, M, N, and their derivatives with respect to u and v. The fundamental magnitudes are not invariant, however, when a new system of parameters u, v is introduced on the surface. Therefore, to decide finally whether two surfaces are congruent or not, it is advantageous, as Lie pointed out (*Leipziger. Berichte*, 1896), to make use of magnitudes which are invariant both under all movements in space, and under all changes of the parameter system on a surface. Such a magnitude is called a differential invariant "in the wider sense": and represents, of course, some geometrical property of the surface at a point (u, v) which is unchanged either by a movement of the surface in space, or by the introduction of a new parameter system on the surface. The principal radii of curvature are seen to be differential invariants of this nature. From this standpoint the author develops in §10 a complete theory of congruence for those surfaces for which no relation of the form

$$W(R_1, R_2) = 0$$

exists between the principal radii of curvature. The latter, the so-called Weingarten-, or W-surfaces, are specially investigated in §11.

In the other sections of this Part, the author discusses the deformation of surfaces. If S' is a surface which is a deformation of a given surface S, that is, if S' is applicable to S, the application of S' to S is equivalent to a point representation of S' upon S, which is conform and which leaves areas invariant. This point representation will not, how-

ever, in general be equivalent to a continuous transformation of S' into S. The skew helicoid, a minimal surface, is an interesting example of a surface which can be transformed into a surface to which it is applicable, the catenoid, by means of a continuous deformation. The original surface is a minimal surface in each of the ∞^1 forms which it assumes under this operation.

In first discussing the problem of deformation, § 3, the minimal curves are used as parameter curves on the surface. It is clear that if S' is applicable to S, the minimal curves of S' will coincide with those of S when the two surfaces are brought to superposition. In order to use the minimal curves as parameter curves, it is necessary to integrate the differential equation $ds^2 = 0$ on both surfaces. If that has been done, and if the expression for ds^2, the square of the arc element, when written in the new variables, can be thrown into the same form on both surfaces, S' is applicable to S.

In § 12, the last section of this Part, the author introduces for the first time the first, second, and mixed differential parameters of a surface; and, following Darboux, as he himself states, completes the discussion of the problem of deformation by showing that the question as to whether S' is a deformation of S can always be determined by mere algebraic eliminations, whenever two independent differential parameters "in the wider sense" of S, or S', are known. It is to be regretted that no example of the application of this theory is given.

The fourth, and last, Part of the volume treats of the theory of the surface curve, by far the greater portion of this Part being devoted to the theory of geodesics. After defining a geodesic, and deducing the differential equation of all geodesics on a given surface, it would have been well to bring out clearly the fact that a geodesic through a point P is definitely determined when the direction of its tangent at P is given: but that the geodesic is not always definitely determined when a second point Q on the curve is given, no matter how near Q may be to P.

The fact that the miminal curves are geodesics is sufficiently emphasized.

In § 2, under the caption of the geodesic representation of one surface upon another, we have the proof of Liouville's important theorem that the geodesics on any surface, for which ds^2 can be thrown into the form

$$ds^2 = (U(u) + V(v))(du^2 + dv^2),$$

can be found by quadratures. All surfaces of constant curvature, and surfaces of revolution, belong, of course, to this class.

After showing that a geodesic point representation of one surface upon another is not always equivalent to a deformation, the author closes this section with a pretty example of the general theory, in which the equations defining the most general geodesic representation of a plane upon itself are derived. They are, of course, the equations of the general projective point transformations in the plane.

The next two sections serve to introduce the Gaussian systems of geodesic coördinates. After proving that the total curvature of a geodesic triangle is equal to the sum of the angles of the triangle, diminished by two right angles, the author proceeds to remark that a geometry of the geodesics on a surface of constant curvature can be developed, which is analogous to that portion of the geometry of the straight line in the plane which does not depend upon the fact that the sum of the three angles of a plane triangle is equal to two right angles. This follows from the fact that the finite equation, in u and v, of the geodesics on a surface of constant curvature, is linear.

The interesting fact, however, is not mentioned that since a surface of constant positive curvature, K, is applicable to a sphere of radius $\frac{1}{\sqrt{K}}$, the formulas of spherical trigonometry, which give the relations between the sides and angles of a spherical triangle, hold also for the sides and angles of any geodesic triangle on the given surface of constant curvature.

A discussion of the geometry of the surface of centers, and the development of the conditions under which the ∞^2 generators of a line complex can be the normals of a surface, occupy §§ 5, 6.

The volume closes with § 7, the curvature and torsion of a general surface curve.

After defining the geodesic, or tangential, curvature, and the normal curvature of a surface curve, the author is careful to point out that the term "geodesic circle" is used by some writers to mean a surface curve of constant geodesic curvature, and by others to mean a surface curve the points of which are at a constant geodesic distance from a fixed point on the surface. The proof is given, in a line or two, that the two definitions are not generally coincident.

The expression for the absolute torsion of a surface curve is also derived; but no mention is made of geodesic torsion,

although this property of the surface curve is very often referred to and used by modern writers upon these subjects.

This last section of the book is very much compressed; but the reader will find there at least the fundamental theorems necessary for a more extensive study of surface curves.

Very few misprints which could mislead the reader have been observed in the volume. One which is, perhaps, worth mentioning is the use of the word "Flächen" for the word "Ebenen" in theorem 7, page 26.

The figures are drawn with care, and are numerous, which is a very helpful feature of the work for a beginner. The references, in footnotes, to other treatises and to original sources, are full and instructive. Tables containing the principal formulas established are appended at the end, as is also an excellent index.

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SOME RECENT BOOKS ON MECHANICS.

The Principles of Mechanics: An Elementary Exposition for Students of Physics. By FREDERICK SLATE. Part I. New York, The Macmillan Co., 1900. Crown 8vo., vii + 299 pp.

A Treatise on Elementary Dynamics. By H. A. ROBERTS. London, Macmillan & Co., 1900. 16mo., xi + 258 pp.

Die Dynamik der Systeme starrer Körper von E. J. ROUTH. Autorisierte deutsche Ausgabe von ADOLF SCHEPP mit einem Vorwort von FELIX KLEIN. Zwei Bände. B. G. Teubner, Leipzig, 1898–1900.

DURING the past decade many works which treat mechanics from one standpoint or another have been published. The list of authors includes Appell, Boltzmann, Föppl, Gray, Hertz, Routh, Volkmann, and others perhaps less well known. The different authors differ greatly both in their aims and in their methods of presenting the subject; but on the whole they show a tendency toward something which is to a considerable extent a recent development—to the careful consideration of the underlying principles of mechanics. The amount of attention given to this phase of

the subject varies from one extreme in the work of Hertz, in which every step from the first to the last is built up with geometric logic by means of more or less artificial devices and points of view, to the other extreme in those works, especially by English authors, in which force and mass are taken as practically innate, *à priori*, ideas and are left with no further refinements of conception than the naïve intuition of the student may supply. In one case the careful examination of the concepts involved usually leads to the result that the fundamental dynamical equation

$$\text{Force} = \text{Mass} \times \text{Acceleration}$$

is merely the definition of either force or mass. In the other case, where force and mass are taken for granted, this equation becomes a physical law or mathematical axiom founded, like Euclid's parallel axiom, upon universal experience.

Notwithstanding that this latter point of view still finds able advocates there is undoubtedly a general trend toward a more thorough discussion of the foundations of mechanics. The laws of Newton, as stated by him and as used for two centuries, no longer satisfy us. This lack of satisfaction is but one of the many similar manifestations of the present state of mathematical instruction and mathematical science. We are no longer content to bear with superficially clear statements which seldom if ever lead into actual error—nor does it suffice to start with inaccurate statements and, as we advance, to modify them so as to bring them into accord with our wider vision and our more stringent requirements. No. We must from the beginning bring up ourselves and our pupils on not only the truth but the whole truth. How soon the recent researches of Hilbert and others on the foundations of geometry must take their place in elementary text-books on plane and solid geometry cannot be said. But that is purely a matter of time unless some reactionary tendency sets in—and there are some who think it already beginning to set in.

At present, however, the treatises and text-books on mechanics are filled with a deal of half scientific, half metaphysical, and generally quite crude discussion of the underlying principles of mechanics. The "principles of mechanics" as denoting the "elements of mechanics" and as denoting the "scientific foundations of mechanics" are becoming synonymous. Thus the task of the reviewer is rendered triply hard, inasmuch as not only must he sift out the good from the bad, but must do this in each of the

various aspects, mathematical, physical, metaphysical, which the author touches upon. In the following series of reviews an attempt will be made to keep each of these aspects separate from the others and still give a connected idea of each work under discussion.

While authors in England, Germany and France have been flooding the market with expositions of all the various branches of mechanics, instructors in our own universities and schools other than technological have not written upon the subject. Works on the mechanics of engineering, graphical statics, strength of materials, and so forth we have in great number; but, with the exception of Professor Ziwet's book, scarcely a work upon theoretical mechanics has been published in this country. What the type of our instruction in theoretical mechanics shall be, whether we shall lean toward the English, the German, or the French, is a question which is not yet settled. It is, however, a question of moment and must inspire at present an exceptional interest in such treatments of mechanics as may come from American authors. But Professor Slate's book needs not rely upon the fact that it is American to awaken interest. It contains numerous novel features and many improvements which will recommend it strongly to teachers and students.

At the commencement of the preface, Professor Slate states that the ideas which have guided the selection and presentation of the material are three: first, to select the subject matter with close reference to the needs of college students; second, to bring the instruction into adjustment with the actual state of their training; and third, to aim continually at treating mechanics as a system of organized thought having a clearly recognizable cultural value. Three better aims could not be chosen. Moreover the students for whom the book is intended are to have had a working knowledge of the calculus and a good ground work of experimental physics. This preparation is ideal, but perhaps more that can fairly be expected of students beginning mechanics. Certain it is, however, that to attempt to teach mechanics as mechanics before the student has had calculus is next to useless and equally certain it is that some previous knowledge of experimental or descriptive physics is a great aid though by no means an absolute necessity in acquiring true mechanical intuition. The preparation Professor Slate expects of his students and the aims which he sets himself are alike harbingers of a merited success for his text-book.

The fundamental idea of a vector is emphasized in connection with velocity and acceleration to the extent of introducing a distinctive notation—a circumflex accent—for vectors. Thus v is the scalar speed; $\hat{v}$, the veetor velocity. The notation is unfortunately clumsy at best and would become excessively so if used consistently throughout the book. Why could not the established custom of using heavier faced type for vectors be followed? The intrinsic value of the emphasis laid upon the idea of a vector nevertheless more than atones for the awkwardness of the accidental notation. In later chapters moments and work are mentioned as the vector and scalar products of forces and distances, although no subsequent use is made of this fact.

The first two chapters, fifty-eight pages, deal with kinematics. The treatment is by no means conventional and is noteworthy in a number of points. From the start there is a clearly defined distinction between the assumed fixed reference system and the coördinate system which may be moving relative to it. Many theorems are proved by geometric or partially geometric instead of by purely analytic methods. This is particularly true of the discussion of instantaneous centers and of accelerations. Perhaps a treatment slightly more analytic would not be amiss. Possibly too the author would do well to give both treatments as he does in several instances. For grasping the meaning of a theorem a geometric proof is better; but for specific applications the analysis is generally necessary. With no further aid than that given in the text a student would have difficulty in determining the space and body centrodes for any given case of motion. It is interesting to note that in the attempt to simplify the difficult subject of acceleration the author begins with the definition of acceleration parallel to a line or component acceleration and afterwards combines the components into the resultant. He then points out the fundamental fact that this resultant acceleration is really nothing but rate of change of velocity regarded as a vector. From a careful perusal of the text it does not appear that this method of treatment succeeds in simplifying matters to any great extent.

The third chapter, entitled Mass and Force, is decidedly the poorest in the book. It can offer few apologies for itself on any score. An attempt is made to discuss mass and force from the "modern" standpoint. The discussion for the most part is extremely crude—a jumble of empirical physics and partially accurate logic—neither clear nor satisfying to logician or practical physicist.

In the first place the author seems to be thoroughly imbued with the idea that kinematics and dynamics are two sciences which differ not in degree alone but in kind; that kinematics is a branch of pure mathematics, but dynamics a branch of physics founded, unlike the former, upon experiment.* Why time and space, equally with inertia, matter, and force, are not derived from universal experience is difficult to see. In fact, speaking from the standpoint of psychology, would it not be true that force, qualitatively at least, earlier than either space or time makes a definite impression upon the individual and does not force always appear more real than either? Perhaps force enters more critically into the struggle for existence. Passing from the qualitative to the quantitative, is it not clear that the so-called parallel axiom of Euclid must be derived in the final test from experience? Experiments by parallactic measurements upon fixed stars have been suggested seriously for making the test. Mass and force may be perceived and defined independently of the truth of this axiom. Non-euclidean mechanics is as possible as non-euclidean geometry. Surely it is not well to lay so great stress on the inherent difference in empiricism between mechanics and geometry. Professor Slate would probably not have done so, had Newton been a contemporary of Euclid.

In the second place Professor Slate, after drawing such a distinct line of demarcation, proceeds with a strange ironical perverseness to obliterate it. One would think this line of demarcation but the natural precursor of the old Newtonian treatment, a justification for assuming mass and force as innate ideas and for stating the postulate that force equals mass times acceleration. But, no!, the following definitions are laid down.†

1° Bodies are said to manifest *inertia* in proportion as it is more difficult to set them in motion.

2° *Force* is said to be acting whenever the physical conditions are such that velocity is changed in magnitude or direction.

3° Two sets of physical conditions are considered as calling into play *forces of equal magnitude* if they are capable of producing in a given body accelerations that are numerically equal. The direction of the force is that of the acceleration which it produces.

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*See pp. 2–3 and p. 59.

†For convenience the definitions are arranged more formally than in the text.

4° The term *mass* is in general use to denote the measured inertia of bodies. The *mass ratio* of two bodies is defined as the inverse ratio of the accelerations producible in them by equal forces.

5° Since, when accelerations are equal, the greater force must be applied to the greater mass and since "it is also accepted in our conception of force" (How? By physics or metaphysics?) "that it increases with the acceleration exhibited in the same body," then "in conformity with these two aspects of the relation—(1) when masses are different and accelerations equal—(2) when accelerations are different and masses equal—force is in fact measured as *proportional* to mass and acceleration conjointly; *i. e.*, to the product of mass and acceleration."

Just what this fifth statement is intended to mean or what standpoint it is intended to represent is nearly a matter for pure conjecture. Professor Slate's none too lucid style becomes unfortunately obscure at this critical point. Then, too, apart from the fact that the first definition naïvely implies an innate knowledge of force in the word *difficult*, it is clear that empiricism plays no rôle here. These definitions are as purely mathematical and free from experiment as if geometric. There is no necessity for previously laying stress on the physical side. When it comes to applying theoretical results to practical experience, a few words concerning the assumed applicability may be in place but not before. A careful examination of Boltzmann's Vorlesungen über die Principe der Mechanik, Leipzig, 1897, will show what may be done in the line of accurate definition in mechanics. That which appears in the book under review is so unfinished and inaccurate as to be valuable in no other way than to call attention to the difficulties involved and give the student a few well chosen illustrations to think over if he finds leisure.

Finally, even if the treatment which Professor Slate has attempted had been carried out in a manner beyond adverse criticism, would his attempting it be unquestionable? Does any such treatment fall in with those three aims he set himself at the start? Not with the first. Not with the second. Not necessarily with the third. Few of the students for whom the book is intended would understand such definitions as given, fewer could appreciate them, fewer still could use them. It would probably have been better, all things considered far better, to have stated the laws of motion in a more customary form; to have explained, discussed, and

illustrated them with that aptness of which Professor Slate is evidently a master; possibly to have pointed out at the end those lines along which modern mathematicians are trying to perfect geometry and mechanics. The students would then have a more distinct conception of the ideas involved in mechanics, a greater ability to make practical use of them, and no less appreciation of "improvements" recently introduced or yet to be introduced. Fortunately, however, there is little need of worrying about the students. They will survive this chapter and disregard it. They invariably feel quite sure that they know what mass and force really are.

Passing on to the remaining chapters of the book, there is little save unconditioned praise to be said. Professor Slate has done that which has long needed to be done. He has shown that the principles of elementary analytical mechanics are few and simple, the applications many and various. Students who use his book will be under the constant necessity of thinking for themselves. It will be useless for them to look for a formula by which each assigned problem may be solved. The unthinking, formula hunting undergraduate will doubtless be perplexed and at a loss at first; but elementary mechanics affords the best of opportunities for learning to think.

In Chapter IV the analytic discussion of the motion of the center of gravity of a system or of a particle, which is the same thing, is reduced to the three fundamental equations—the force equation, the momentum equation, the work equation. In Chapter V these results are extended to rotation about a fixed axis and to uniplanar motion in the most natural and simple manner, merely by treating a rigid body as made up of an infinite number of elements dm to each of which the foregoing equations apply. Later in the chapter, motion and systems of forces in three dimensions are touched upon. Chapter VI deals with harmonic motions and pendulums. The discussion is elaborate, including damped and undamped vibrations, simple, compound, torsion, and bifilar pendulums. An excellent collection of thought existing problems concludes the chapter. Chapter VII treats the law of the inverse square, potential, and orbits described under the attraction due to gravitation. Chapter VIII exhibits the application of the three fundamental equations to a vast number of dissimilar problems. Chapter IX takes up the theory of dimensions together with the C.G.S. and weight system of units. The concluding Chapter X discusses moments of inertia and

centers of gravity. One hundred and sixty-three exercises which are well chosen if not a trifle too difficult and an extensive index close the volume.

In conclusion we would say that with the exception of the third chapter the book carries out the three aims of its author to our entire satisfaction. We would recommend it most heartily to all who wish to think themselves into the heart of mechanics with as little purely formal analysis as possible. Teachers will probably find that as they teach the book they will have to assign numerous problems easier than those inserted at the end of the chapters. This, however, is a small matter. We are glad to notice that Professor Slate is planning a second part to his work which will treat mechanics in three dimensions. We wish him the best of success, and hope that he will not delay the publication of this sequel.

Mr. Roberts's Treatise on Elementary Dynamics is one of the multitudinous English text-books which attempt to teach mechanics without the differential and integral calculus, generally without analytical geometry, often even without trigonometry. In this case a knowledge of trigonometry and the elements of analytical geometry is assumed. This book is therefore as advanced as any of its kind can be. In numerous places the notation of the calculus is also given for the benefit of those students who may be beginning that subject. The treatment is excellent from first to last. Addition and subtraction of vectors, (vector) velocities, (vector) accelerations, angular velocities, the laws of motion including a short statement of the "modern" point of view, motion under gravity and other simple accelerations such as that afforded by elastic strings, work and energy, potential energy and conservative systems, dimensions, direct and oblique impact, projectiles, cycloidal and simple pendulums are among the subjects discussed.

The book may be of great use in England; but to what class of pupils it may be of service in this country does not appear. We seldom teach mechanics between analytical geometry and calculus. Moreover this treatise looks too difficult, too much extended, for those who have not had calculus. It would, however, be very serviceable to such as were studying elementary mechanics by means of the calculus, but found that the none too familiar analytic methods created so much confusion and distraction as to render the ideas involved in mechanics unreal and vague.

Even students of considerable advancement in mechanics might do well to read this little book for the sake of the perspective which they might thus acquire.

Dr. Routh's treatises upon the various branches of mechanics, statics, dynamics of a particle, stability of motion, and in particular rigid dynamics, are so well known as to need no notice. It is therefore not surprising that Teubner, who is ever ready to publish a German translation of the best scientific literature of all nations, should now print this edition of Dr. Routh's Rigid Dynamics. As Professor Klein points out in his preface, this work is wholly different from any which has previously been available in German. To teach mechanics even in the most advanced portions from the standpoint of solving problems is furthest from the German method. We remember one instance in which a German reviewer recently said of a work under review that it was remarkable for its numerous examples. A count showed not more than forty in about five hundred pages of text. How Dr. Routh's work impresses this reviewer may be difficult to imagine. These English books are, however, an extreme. Placing such emphasis on the solution of problems is a result of the system of examinations at the colleges. The student is too apt to lose his perspective and to forget what the *theory* of mechanics is. In this the Germans are far ahead. We daresay that our English publishers would render no less service to our own scientific literature by translating the best German presentations of mechanics into English than Teubner has now rendered to German scientific literature by his translation of Dr. Routh's classic treatise on rigid dynamics.

EDWIN BIDWELL WILSON.

YALE UNIVERSITY,
March 30, 1902.

THE GALOIS THEORY IN BURNSIDE AND PANTON'S THEORY OF EQUATIONS.

ONE of the most welcome additions to Burnside and Panton's "Theory of Equations" is the appearance in the new edition (the fourth) of a chapter devoted to the theory of substitutions and the theory of equations from the Galois standpoint. The British interest in the methods of Galois never has been very deep and about all the national literature is comprised in the last two pages of Cayley's article "Equation" in the Encyclopedia Britannica and four or

five of his shortest papers, together with a few memoirs on special topics by other writers (as *e. g.*, Kirkman), while the present work is the first orderly presentation of the subject that has been undertaken by British mathematicians.

The authors begin with a discussion—modelled after Serret—of the general properties of substitutions and of multivalued functions. Everything is clearly and accurately written, and the authors are successful in the difficult task of making this rather dry subject interesting reading. The successive concepts are introduced in a natural manner, and the wording of both theorems and definitions is exact and careful. The numerous examples will supply a much-felt need in enabling the student to obtain a good foothold in the fundamentals of the theory.

When these preliminary developments have been completed, the group of an equation is defined and its existence established. Here, however, the treatment becomes open to serious criticism. For the all-important distinction between the formal and the numerical invariance of a function is disregarded and this omission weakens and vitiates the whole presentation. The matter is sufficiently familiar. Two formally distinct values of a function of the roots of a given equation may be numerically equal, and then the discriminant of the resolvent equation vanishes. This renders impossible the formation of the group, as defined by Burnside and Panton, and other methods must be adopted. The oversight is rather surprising, moreover, since nearly every writer on the subject (including Galois, Jordan, and Serret) is careful to say "numerical value" when speaking of the group—I believe the only exceptions are Netto, Hagen, and Pascal. The error in the "Repertorio" of the latter is corrected by Schepp in his German version of the book, while on this point Netto's treatise has undergone a more thorough and almost microscopically exhaustive series of reviews than falls to the lot of most mathematical productions. In fact Vogt, in his seventh chapter, goes to the trouble of re-proving the formal invariance theorems again *de novo* for numerical invariance, even committing the opposite error of stating that a group is formed by all the substitutions that leave a function numerically invariant. Of the same kind is a misstatement on page 246 of Burnside and Panton's work, where the coefficients of the Galois function are taken (following Netto) as "distinct arbitrary constants." Inasmuch as there is an infinite series of values that these coefficients cannot assume, the word "arbitrary" should be omitted; Kronecker's "distinct indeterminates" are something quite different.

The relation between the group and the equation is illustrated by the elementary equations as examples, and the irresolvability of the general equation is then discussed by Wantzel's (or Ruffini's*) method, slightly modified and simplified from Vogt's form. But inasmuch as Burnside and Panton have already introduced the group of the equation, it would have been preferable to use a more modern proof, resting on the properties of that group. By this method a somewhat tedious analysis is avoided, and the important connection between the group and the domain of rationality is exhibited. As the discussion stands few students are likely to appreciate this connection.

The brevity of the chapter and the consequent exclusion of many immediately connected topics of great interest is a matter of regret. Among the chief of these topics are the distinction between simple and composite groups (barely referred to on page 244), the question of primitivity and imprimitivity, the general theory of resolvents, and the properties of abelian equations (uniserial abelians are treated in an appendix). A word or two on Galoisian equations and the general solvable equations would not have been amiss. Personally I think it would have been better to have omitted certain details in other parts of the book in order to make room for a more complete treatment here—supposing that the authors did not care to undertake such a radical change as the early introduction and the systematic employment of the Galois principles, advantageous though the change would have been.

One last criticism. The references given—Serret, Jordan, Netto and Vogt—are not very complete. Weber's "Lehrbuch der Algebra," which is past question the most important work on the subject, is not mentioned. Hölder's very valuable monograph in the Encyclopädie der mathematischen Wissenschaften, Netto's Algebra, and Bianchi's "Lezioni sulla teoria dei gruppi di sostituzioni" certainly should be also included. The omission of Echegaray's "Lecciones sobre la resolución de las ecuaciones" is more pardonable, and the reprint of Pierpont's articles from the *Annals of Mathematics* was probably of too recent a date to have attracted the attention of the writers.

BURTON SCOTT EASTON.

UNIVERSITY OF PENNSYLVANIA,
April 3, 1902.

* Wantzel's paper in Vol. 4 of the first series of the *Nouvelles Annales de Mathématiques* (1845) is the source to which this proof is commonly referred, but substantially the same was given by Ruffini in 1813 in his book "Reflessioni intorno alla soluzione dell'equazioni algebraiche."

SHORTER NOTICES.

Karl Friedrich Gauss:—General Investigations of Curved Surfaces of 1827 and 1825. Translated with notes and a bibliography by JAMES C. MOREHEAD, A.M., M.S., and ADAM M. HILTEBEITEL, A.M., J. S. K. Fellows in Mathematics in Princeton University. Princeton, The Princeton University Library, 1902. 4to., viii + 127 pp.

THE handsome form of this the first translation into English of these classic memoirs of Gauss may make amends for its tardy appearance. Printed on old Stratford paper, gilt top, uncut edges, under press work that leaves little to be desired, the volume is one to delight both bibliophile and mathematician.

The translation proper is preceded by an introduction written by Professor H. D. Thompson which discusses the evolution of the method of Gauss as exhibited in the memoirs, and presents what is believed to be a complete account of all previous translations and editions.

The translation has been executed with great care from the original text of each memoir, free use having been made of other editions, especially that of Professor A. Wangerin* of the memoir of 1827. The translators have endeavored to retain as far as possible the notation, the form and punctuation of the formulæ, and the general style of the original papers. Certain changes necessary to conform to more modern notations are mentioned in the notes. These notes, which occupy thirty-two pages of the book, contain proofs of many of the formulæ and theorems which Gauss only indicated without demonstration.

The bibliography consists of three hundred and forty-three titles, and, with a few exceptions, is limited to books, memoirs, and papers which use the method of Gauss and which treat generally or in detail one or more of the following subjects: curvilinear coördinates, geodesic and isometric lines, curvature of surfaces, deformation of surfaces, orthogonal systems, and the general theory of surfaces.

The publication of the translation was made possible by the liberality of the Princeton Library Publishing Association and the alumni of the university who founded the mathematical seminary.

E. O. LOVETT.

* Ostwald's Klassiker der exakten Wissenschaften, No. 5.

Histoire des Mathématiques dans l'Antiquité et le Moyen Age. H. G. ZEUTHEN. Edition Française, revue et corrigeé par l'auteur, traduite par JE·N MASCART. Paris, Gauthier-Villars, 1902. Pp. xvi + 296.

THIS work originally appeared in Danish in 1893. Two years later a German translation was published, thus bringing it quite generally before the mathematical world. It is a compliment to the excellence of the work that now a French edition should make it known to those who confine their studies to the Latin tongues Furthermore it is a fortunate circumstance that Professor Paul Tannery has taken such interest in the translation as to examine the manuscript and append a number of valuable notes. Among these notes is the record of the comparatively recent discovery of the Metrics of Heron.

The size of the work is some evidence of its nature. It makes no pretense to set forth original discoveries. It claims merely to digest the important facts as presented by the works of Cantor, Hankel, Chasles and others of their class. These facts it seeks to place in relief, neglecting as unessential to the purpose in view the minor details, the question of the discovery of various theorems, and the controversies over dates and incidents of biography. The result is a manual whose usefulness is not limited merely to the needs of teachers of mathematics in Denmark, for whom it was written, and who, by a wise regulation of the government, are required to pass an examination upon the history of the subject.

The work is confined in large measure to the history of Greek mathematics, about three fourths of the space being thus employed. Less than ten per cent of the space is given to Hindu mathematics, and about fifteen per cent to the middle ages, including the work of the Arabs. This allotment is probably a fair one in the present state of our knowledge, and is certainly one to be expected from a man who has done so much in the study of Greek mathematics. Suter's recent investigation of the mathematical literature of the Arabs, and Braunmühl's exposition of their contributions to trigonometry, are, however, likely to change the common estimate that they were merely transmitters of knowledge, and to give them more prominence in future works.

For so brief a sketch, the work is to be commended for its attempt in several instances to bring out causal relations and to evaluate the work of prominent writers. This is

best shown in the discussion of the state of mathematics before Euclid, and of the Elements of this writer. The weakness of the definitions and postulates as set forth in this classic work are treated in such a way as to furnish abundant material for thought on the part of the champions of Euclid as a text-book writer. The note which follows this discussion, Sur les hypothèses de la géométrie (Anmerkung über die Voraussetzungen der Geometrie), is a good résumé of the question of non-euclidean geometry.

The work is suggestive in its references to the connection between the ancient and modern mathematics. This appears in the discussions of the quadratures of Archimedes and their relation to the integral calculus. In view of the author's high appreciation of the fifth book of Euclid, it is a little strange that the relation of this to the modern theory of irrationals is not mentioned.

Since the work is merely a sketch, it would be unfair to enter upon any extended criticism of details, although there is frequently a temptation to do so. This temptation is particularly strong where the author's statement is true and yet conveys a false impression. Thus after speaking of Widman's arithmetic (or Widmann's, as the name appears here, and usually in later works), he says: "Dès 1483, d'autre part, avait été imprimée *l'Arithmétique* dite *de Bamberg.*" This is true, but it conveys the impression that this was the first printed arithmetic, at least in Germany, when it was neither the first nor the first German one to appear. The title of Widman's book looks odd in French translation, Calcul adroit et joli pour tout commerce; but the German title looks still more peculiar, since it is neither the original, Behend und hüpsch Rechnung vff allen Kaufmanschaften (1508 edition, the others varying slightly), nor the modern form. The statement is made that the symbols + and − are used in this work, a true but a misleading one. It is true that Widman says (edition of 1508, fol. 59v.), "Was − ist das ist minus * * * vnd das + das ist mer," but he uses the symbols not to express operations but to indicate excess or deficiency, as 4 cwt. + 5 lbs., 4 cwt. − 17 lbs. A similarly misleading statement is made in connection with Fibonacci, who, it is said, "est originaire de Pise, importante ville commerciale, *où*, de bonne heure, il apprit le calcul sur l'abaque." As a matter of fact the evidence is in favor of his having learned number and the abacus at Bougie, where his father was a factor, a point of much interest in the history of his *Liber abaci.*

The salient features of the work are, however, commendable, and it is hoped that its publication in France may awaken new interest in a phase of the subject in which that country at one time excelled all others.

DAVID EUGENE SMITH.

NOTES.

THE Ninth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at Northwestern University, Evanston, Ill., in the first week of September. A preliminary circular in regard to the meeting will be issued about May 20th.

THE first meeting of the Pacific Section of the AMERICAN MATHEMATICAL SOCIETY was held at San Francisco, May 3. A report of the meeting will appear in a later number of the BULLETIN.

THE Librarian acknowledges the gift to the AMERICAN MATHEMATICAL SOCIETY of six volumes of mathematical works presented by M. C. Naud, Paris, and one volume by Delalain Frères, Paris.

THE seventy-fourth general meeting of the Gesellschaft Deutscher Naturforscher und Aerzte will be held at Karlsbad, September 21–27. Section I includes mathematics, astronomy and geodesy; section III, applied mathematics and physics.

THE twenty-third general meeting of the Deutscher Geometer-Verein will be held at Düsseldorf, July 20–23.

THE annual stated meeting of the National Academy of Sciences was held at Washington, D. C., April 15–17. Among the papers presented was one on "The postulates of geometry," by Mr. C. S. PEIRCE.

THE second (April) number of volume 24 of the *American Journal of Mathematics* contains the following articles: "Canonical form of a linear homogeneous transformation in an arbitrary realm of rationality," by L. E. DICKSON; "A new theory of collineations and their Lie groups," by

H. B. NEWSON; "Infinitesimal deformation of surfaces," by L. P. EISENHART.

THE April number (second series, volume 3, number 3) of the *Annals of Mathematics* contains: "Space of constant curvature (conclusion)," by F. S. WOODS; "Brilliant points and loci of brilliant points," by W. H. ROEVER; "Problems in infinite series and definite integrals, with a statement of certain sufficient conditions which are fundamental in the theory of definite integrals," by W. F. OSGOOD; "Note on the product of linear substitutions," by H. B. NEWSON.

THE *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, founded by Dr. J. C. V. HOFFMANN in 1869, will be edited in the future by Dr. H. SCHOTTEN, assisted by a board of twenty-three collaborators. The new form is a larger octavo than that of the former series and is now uniform with the other mathematical publications of the Teubner press. The last number, the first of the thirty-third volume, is accompanied by a portrait of Dr. HOFFMANN and an account of his services to secondary education in Germany.

UNIVERSITY OF CHICAGO.—The following advanced courses in pure and applied mathematics are announced for the summer quarter.—By Professor E. H. MOORE: Differential calculus, five hours; Foundations of geometry, five hours.—By Professor H. MASCHKE: Higher plane curves, five hours; Elliptic functions, five hours.—By Professor L. E. DICKSON: Theory of substitutions, five hours.—By Mr. A. C. LUNN: Advanced integral calculus.—By Mr. O. VEBLEN: Differential equations.—By Dr. F. R. MOULTON: The problem of three bodies.

HARVARD UNIVERSITY.—The following advanced mathematical courses are offered during the academic year 1902–1903:—By Professor J. M. PIERCE: Triangular coördinates and algebraic plane curves, especially cubics; Quaternions (second course); † Linear associative algebra; † Algebra of logic.—By Professor W. E. BYERLY: † Dynamics of a rigid body; Trigonometric series, spherical harmonics, and potential function.—By Professor W. F. OSGOOD: Calculus (second course); † Algebra, Galois's theory of equations; Theory of functions (first course); † Theory of functions (advanced course), transcendental integral and fractional functions.—By Professor M. BÔCHER: Modern geometry; † Infinite series and products; † Introduction to partial

differential equations; † Theory of functions (advanced course), definite integrals.—By Dr. C. L. BOUTON: Differential equations, with introduction to Lie's theory of continuous groups.—By Mr. J. K. WHITTEMORE: † Differential geometry of curves and surfaces; Hydrostatics, hydrokinematics, hydrokinetics.—By Mr. J. L. COOLIDGE; † Theory of equations, invariants; Geometry of position.

These courses will involve three lectures a week throughout the year, except those preceded by †, which involve about half this number of lectures. Professors OSGOOD and BÔCHER, Dr. BOUTON and Mr. COOLIDGE also offer courses in reading and research on Theory of functions, Theory of differential equations, Theory of continuous groups, and Geometry, respectively.

The mathematical conference will meet twice a month.

The several foreign universities below offer during the summer semester of the current academic year, courses in mathematics as follows:

UNIVERSITY OF BERLIN.—By Professor L. FUCHS: Introduction to the theory of functions, four hours; Introduction to the theory of differential equations, four hours, seminar.—By Professor H. A. SCHWARZ: Synthetic geometry, four hours; Theory of elliptic functions, four hours; Theory of analytic functions, two hours; Colloquium, two hours; Seminar.—By Professor G. FROBENIUS: Analytical geometry, four hours; Seminar.—By Professor K. HENSEL: Differential calculus, four hours; Higher theory of numbers, four hours; Axioms of geometry, two hours.—By Professor J. KNOBLAUCH: Integral calculus, four hours; Approximate calculation of definite integrals, one hour; Theory and application of determinants, four hours.—By Professor G. HETTNER: Infinite series, products and continued fractions, two hours.—By Professor R. LEHMANN-FILHÉS: Analytical mechanics, four hours; Exercises in mechanics, one hour.—By Dr. E. LANDAU: Higher algebra four hours, with exercises, two hours; Determination of class numbers of binary quadratic forms.

UNIVERSITY OF BONN.—By Professor R. LIPSCHITZ: Elements of differential and integral calculus, four hours; Exercises in seminar, two hours.—By Professor H. KORTUM: Theory of functions, four hours; Infinite series, two hours; Exercises in seminar, two hours.—By Professor L. HEFFTER: Theory of linear differential equations, four hours; Descriptive geometry, with exercises, five hours.

UNIVERSITY OF HALLE-WITTENBERG.—By Professor G. CANTOR: Selected chapters from the theory of elliptic functions, two hours; Higher algebra, three hours; Seminar, two hours fortnightly.—By Professor A. WANGERIN: Differential calculus, with exercises, five hours; Theory of space curves and surfaces, five hours; Analytical mechanics, II, one hour; Seminar, two hours fortnightly.—By Professor V. EBERHARD: Higher algebraic curves, two hours; Theory of invariants, three hours.

UNIVERSITY OF INNSBRUCK.—By Professor O. STOLZ: Real differential and integral calculus, four hours; Complex numbers and introduction to the general theory of functions, three hours.—By Professor W. WIRTINGER: Algebraic functions and their integrals (continuation), five hours; Seminar, two hours.—By Dr. K. ZINDLER: Differential equations, three hours; Selected chapters of elementary mathematics for students of natural science, two hours; Seminar, one hour.

UNIVERSITY OF KIEL.—By Professor L. POCHHAMMER: Analytical geometry of space, four hours; Selected chapters of the theory of partial differential equations, four hours; Seminar, one hour.—By Professor P. HARZER: Selected chapters of celestial mechanics, three hours.—By Professor P. STÄCKEL: Differential calculus and introduction to analysis, four hours; Algebraic curves and surfaces, four hours; Life and work of Abel, one hour; Seminar in the algebraic memoirs of Abel.

UNIVERSITY OF KÖNIGSBERG.—By Professor F. MEYER: Differential calculus, three hours, with exercises, one hour; Higher algebra, four hours; Seminar, one hour.—By Professor A. SCHOENFLIES: Elliptic functions, four hours; Seminar, two hours.—By Professor L. SAALSCHÜTZ: Bernouilli's numbers, two hours; Theory of definite integrals, two hours.

UNIVERSITY OF LEIPSIC.—Professor W. SCHEIBNER offers no lectures and those of Professor A. MAYER may be announced later.—By Professor C. NEUMANN; Constructive geometry, in particular conic sections. four hours; Seminar, two hours.—By Professor O. HÖLDER: General theory of functions of a complex variable, two hours; Theory of numbers, two hours; Seminar, one hour.—By Professor F. ENGEL: Application of the differential and integral cal-

culus to geometry, four hours; Introduction to the theory of transformation groups, two hours; Transformation groups and the theory of invariants, one hour; Seminar, one hour: exercises in the theory of groups, one hour.—By Professor F. HAUSDORFF: Ordinary differential equations, four hours, with exercises, one hour; Non-euclidean geometry, two hours.—By Dr. H. LIEBMANN: Plane analytical geometry, four hours, with exercises, one hour; Graphical statics, two hours.

UNIVERSITY OF PRAGUE.—By Professor G. PICK: Algebraic equations, three hours; Algebraic functions, two hours; Seminar, two hours.—By Professor J. A. GMEINER: Differential and integral calculus, four hours; Introduction to the theory of functions of a complex variable, one hour.

UNIVERSITY OF VIENNA.—By Professor G. v. ESCHERICH: Theory of functions, five hours; Proseminar, one hour; Seminar, two hours; Theory of probability, three hours.—By Professor L GEGENBAUER: Elements of differential and integral calculus (continuation), five hours, with exercises, one hour; Proseminar, one hour; Seminar, two hours.—By Professor F. MERTENS: Algebra (continuation), five hours; Seminar, two hours; Proseminar, one hour; Mathematical statistics, three hours.—By Professor G. KOHN: Analytical geometry (continuation), four hours, with exercises, one hour; Plane algebraic curves, two hours.—By Dr. K. ZSIGMONDY: Surfaces of second order, one hour.—By Dr. D. v. STERNECK: Theory of numbers, four hours.—By Dr. K. CARDA: Applications of the theory of groups to integration problems and infinitesimal geometry.

A CIRCULAR recently sent out by the "Commission du Répertoire Bibliographique des Sciences Mathématiques" gives some information concerning the actual state of progress of this enterprise. The object in view is a complete subject catalogue of all mathematical periodicals of the 19th century. The classification used is that of the "Index du répertoire bibliographique," of which an abstract is found in every number of the *Revue semestrielle*. The catalogue is made up of slips, 14 x 8.5 cm., each containing about ten titles and marked at the top with the symbol (such as A1a, R8d, etc.) indicating the subdivision of the "Index" to which the memoirs and papers recorded on the slip belong. These slips are sold in sets of 100; eleven such sets, comprising about 1,100 titles, are now ready and can be obtained

from the publishers, Messrs. Gauthier-Villars, of Paris, at the price of 2 francs a set. The Commission, of which M. Poincaré is the president, C.-A. Laisant the permanent secretary, has engaged the collaboration of mathematicians of all civilized nations to compile lists of titels. Among the few countries that have so far contributed nothing we find Great Britain, Greece, Sweden and the United States. The delay in getting contributions from this country is probably due to the recent decease of Professor Craig; Professor J. H. Gore, Columbian University, Washington, D. C., is now in charge of the work for the United States.

THE German Bureau for international bibliography in Berlin began last autumn the publication of a bibliography of the German mathematical and scientific literature. The bibliography appears weekly, and is arranged according to the plan adopted for the international scientific catalogue.

DR. B. S. EASTON, of the University of Pennsylvania, has in press a bibliography of the theory of substitutions, comprising about five hundred and eighty titles with a collection of theorems and definitions.

THE University of Christiana is preparing to celebrate, September 5–7, the hundredth anniversary of the birth of N. H. ABEL.

THE Royal Institute of Lombardy offers a prize of 1200 lire for the best contribution to or noteworthy and orig. al improvement in the theory of groups of transformations. Papers may be written in Italian, French, or Latin, and must be presented anonymously not later than March 31, 1903.

PROFESSOR H. W. TYLER has been appointed head of the mathematical department of the Massachusetts Institute of Technology.

A LABORATORY is to be founded at the University of Edinburg in honor of the late Professor P. G. TAIT.

RECENT catalogues of second hand mathematical works: Oscar Schack, Königstrasse 15, Leipzig, Verzeichniss 95, 445 titles on the theory and history of mathematics and physics.—Raffaello Ginsti, Livorno, Italy, catalogue No. 19, 289 titles.—A. Thury, 5 Rue Petitot Geneva, Switzerland, catalogue 10, 1018 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADHÉMAR (R. D'). See BOREL (E.).

BOREL (E.). Leçons sur les séries à termes positifs, professées au Collège de France. Recueillies et rédigées par R. d'Adhémar. Paris, Gauthier-Villars, 1902. 8vo. 6+95 pp. (Nouvelles leçons sur la théorie des fonctions.) Fr. 3.50

DICKSTEIN (S.). See PASCAL (E.).

DINO (N. S.). See SALMON (G.).

GRAF (J. H.). See SCHLÄFLI (L.).

KLEIN (F.). Zur Schraubentheorie von Sir Robert Ball. 8vo. (*Zeitschrift für Mathematik und Physik*, 47, pp. 237–265.)

LOPUSZANSKI (T.). See SCHUR (F.).

PASCAL (E.). Repertorio delle matematiche superiori. Polish translation by S. Dickstein. Vol. II: Geometry. Warsaw, 1901. 8vo. 10+728 pp. M. 15.00

SALMON (G.). Trattato analitico delle sezioni coniche, contenente un cenno dei più importanti metodi moderni algebrici e geometrici. Versione italiana da N. S. Dino. 6a edizione. Napoli, 1902. 8vo. 641 pp. Fr. 8.50

SCHLÄFLI (L.). Theorie der vielfachen Kontinuität. Herausgegeben von J. H. Graf. Zürich, 1901. 4to. 4+239 pp. M. 12.00

SCHUR (F.). Podrecznik geometryi analitycznéj. Przeklad z niemieckiego, przez T. Lopuszanskiego. Warszawa, 1901. 8vo. 249 pp. M. 3.00

VIVANTI (G.). Corso di calcolo infinitesimale. Ristampa. Messina, 1901. 8vo. 584 pp. Fr. 8.50

II. ELEMENTARY MATHEMATICS.

AMERIGO (L. B.). Lezioncine di aritmetica e geometria ad uso delle cinque classi delle scuole elementari, arricchite di molti esempi pratici e problemi, con soluzione ragionata. 13a ristampa. Genova, Lanata, 1902. 16mo. 78 pp. Fr. 0.60

BALTIN (R.) und MAIWALD (W.). Kurzgefasstes Lehrbuch der Mathematik für Seminare und Präparandenanstalten. Unter Zugrundelegung des Lehrbuchs von H. Müller: Die Mathematik auf den Gymnasien und Realschulen, Teil I, B, nach den Lehrplänen von 1901 für Seminare u. s. w. bearbeitet. Leipzig, Teubner, 1902. 8+214 pp. Cloth. M. 3.00

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